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Universal Functions on Complex General Linear Groups

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In 1929, Birkhoff proved the existence of an entire function F on $\mathbb{C}$ with the property that for any entire function f there exists a sequence $\{a_k\}$ of complex numbers such that $\{F(\zeta + a_k)\}$ converges to $f(\zeta)$ uniformly on compact sets. Luh proved a variant of Birkhoff's theorem and the second author proved a theorem analogous to that of Luh for the multiplicative group $\mathbb{C}^*$. In this paper extensions of the above results to the multi-dimensional case are proved. Let $M(n, \mathbb{C})$ be the set of all square matrices of degree n with complex coefficients, and let $G = GL(n, \mathbb{C})$ be the general linear group of degree n over $\mathbb{C}$. We denote by $\mathscr{A}(G)$ the set of all holomorphic functions on G. Similarly, we define $\mathscr{A}(\mathbb{C})$. Let $\hat{K}$ be the $\mathscr{A}(G)$-hull of a compact set K in G. Finally we denote by $B(G)$ the set of all compact subsets K of G with $\hat{K} = K$ such that there exists a holomorphic function f on $M(n, \mathbb{C})$ with $f(0) \notin (f(K))^\wedge$, where $(f(K))^\wedge$ is the $\mathscr{A}(\mathbb{C})$-hull of $f(K)$. Our main result is the following. There exists a holomorphic function F on G such that for any $K \in B(G)$, for any function f holomorphic in some neighbourhood of K, and for any $\varepsilon > 0$, there exists $C \in G$ with $\max_{Z \in K} |F(CZ) - f(Z)| < \varepsilon$.

0. INTRODUCTION

In 1929, Birkhoff proved the following theorem [5].

THEOREM. *There is an entire function $F(\zeta)$ in one variable such that for any entire function $f(\zeta)$ there exists a sequence $\{a_n\}$ of complex numbers with*

$$f(\zeta) = \lim_{n \to \infty} F(\zeta + a_n) \qquad \text{on } \mathbb{C},$$

where the convergence is uniformly on compact sets.

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Such a function is said to be universal.

There is an extensive literature on the field of functions which are "universal" in different respects. In 1941, Seidel and Walsh [24] showed the existence of universal functions for non-Euclidean translations on the unit disk. Heins [12] considered universal functions in the family of bounded holomorphic functions of modulus at most one on the unit disk. Voronin [25] showed that Riemann's zeta-function has a kind of "universality property." There are extensions and variants of this result (Reich [21, 22], Gavrilov and Kanatnikov [8], and Bagchi [2]).

Luh [16] proved the following theorem in 1976.

THEOREM. *Let $\{a_n\}$ be a sequence in $\mathbb{C}$ with limit ∞. Then there is an entire function $F(\zeta)$ such that for every compact set K with connected complement in $\mathbb{C}$ and for every function $f(\zeta)$ holomorphic in the interior of K and continuous on K, there exists a subsequence $\{a_{n_k}\}$ such that $\{F(\zeta + a_{n_k})\}$ converges to $f(\zeta)$ uniformly on K.*

In 1988 the second author [26] replaced the additive group of complex numbers $\mathbb{C}$ by the multiplicative group $\mathbb{C}^*$ and proved the existence of universal functions on $\mathbb{C}^*$. His theorem is as follows.

THEOREM. *There exists a holomorphic function F on $\mathbb{C}^*$ such that for any compact subset K whose complement in $\mathbb{C}^*$ is connected, for any continuous function f on K which is holomorphic in the interior of K, and for any $\varepsilon > 0$, there exists $c \in \mathbb{C}^*$ with*

$$\max_{\zeta \in K} |F(c\zeta) - f(\zeta)| < \varepsilon.$$

For further results and related problems, we refer to [3, 4, 7, 9, 11, 15, 17–20] and their references.

All of the above results are in the case of one variable. The first author extended the results of Birkhoff and Luh to the case of several variables under a special condition on sequences in $\mathbb{C}^n$ in [1]. Later, the second author pointed out that this condition can be removed.

In this paper, we show that the extensions of Birkhoff's result and Luh's result hold without this condition. In [1], the first author considered multiplicative universal functions in several variables. Let $GL(n, \mathbb{C})$ be the group of all non-singular matrices of degree n with complex coefficients, which is called the general linear group of degree n over $\mathbb{C}$. He proposed the following problem [1].

Problem. Is there a holomorphic function on $GL(n, \mathbb{C})$ which has a kind of universality property as in the case of $\mathbb{C}^*$?

The main purpose of this paper is to answer the above problem. Theorem 4 in Section 3 is our main result.

We should note that there are two papers [6, 10] in addition to [1] that deal with universal functions in several variables. In [10] an extension of Birkhoff's theorem was proved.

1. SEPARATION LEMMA

Let S be a Stein manifold. We refer the reader who is not familiar with Stein manifolds to [13]. We denote by $\mathscr{A}(S)$ the set of all holomorphic functions on S. For a compact set $K \subset S$, we define the $\mathscr{A}(S)$-hull $\hat{K}$ of K by

$$\hat{K} := \{x \in S; |f(x)| \leqslant \sup_K |f| \text{ if } f \in \mathscr{A}(S)\}.$$

A compact set K is said to be $\mathscr{A}(S)$-convex if $\hat{K} = K$.

If $S = \mathbb{C}$, then $\hat{K}$ is just the union of K and relatively compact components of $\mathbb{C} \backslash K$. Therefore, if K_1 and K_2 are disjoint and $\mathscr{A}(\mathbb{C})$-convex, then $(K_1 \cup K_2)^\wedge = \hat{K}_1 \cup \hat{K}_2$. Moreover, any finite disjoint union of $\mathscr{A}(\mathbb{C})$-convex sets is again $\mathscr{A}(\mathbb{C})$-convex.

On the other hand, in the several complex setting there is no such nice geometric characterization of the $\mathscr{A}(S)$-hull of a compact subset K. This is a crucial point when we deal with approximation problems in several variables. However, we have the following lemma which is useful in our arguments. Although Kallin [14] only proved the case in which $S = \mathbb{C}^n$ and f is a polynomial, her proof also works in the more general situation. We give it for the convenience of the readers.

SEPARATION LEMMA [14]. *Let S be a Stein manifold, and let X_1, X_2 be compact subsets of S. Suppose that there exists $f \in \mathscr{A}(S)$ such that*

$$f(X_1)^\wedge \cap f(X_2)^\wedge = \varnothing,$$

where $f(X_i)^\wedge$ is the $\mathscr{A}(\mathbb{C})$-hull of $f(X_i)$ for $i = 1, 2$. Then we have

$$(X_1 \cup X_2)^\wedge = \hat{X}_1 \cup \hat{X}_2.$$

Proof. By the definition of $\mathscr{A}(S)$-hull, it is obvious that $(X_1 \cup X_2)^\wedge \supset \hat{X}_1 \cup \hat{X}_2$. Then it suffices to show that if $x \notin \hat{X}_1 \cup \hat{X}_2$, then $x \notin (X_1 \cup X_2)^\wedge$.

Suppose $x \notin \hat{X}_1 \cup \hat{X}_2$. If $f(x) \notin f(X_1)^\wedge \cup f(X_2)^\wedge$, then there exists $g \in \mathscr{A}(\mathbb{C})$ by Mergelyan's theorem such that $|g(f(x))| > 1$ and $|g(z)| < 1$ on $f(X_1)^\wedge \cup f(X_2)^\wedge$. Letting $h := g \circ f \in \mathscr{A}(S)$, we obtain $|h(x)| > \max_{y \in X_1 \cup X_2} |h(y)|$. Hence $x \notin (X_1 \cup X_2)^\wedge$.

Consider the case that $f(x)\in f(X_1)^\wedge \cup f(X_2)^\wedge$. We may assume that $f(x)\in f(X_1)^\wedge$. Since $x\notin \hat{X}_1$, there exists $g\in \mathscr{A}(S)$ such that $g(x)=1$, $|g|<\frac{1}{2}$ on X_1. Let $M:=\max\{|g(y)|;\ y\in X_2\}$. Using Mergelyan's theorem again, we can take $r\in\mathscr{A}(\mathbb{C})$ such that $|r-1|<\frac{1}{3}$ on $f(X_1)^\wedge$ and $|r|<1/2M$ on $f(X_2)^\wedge$. Then $h:=(r\circ f)\cdot g$ is a holomorphic function on S satisfying that $|h(x)-1|<\frac{1}{3}$, $|h|<\frac{2}{3}$ on $X_1\cup X_2$. Therefore $x\notin (X_1\cup X_2)^\wedge$. ∎

COROLLARY [14]. *If X_1 and X_2 are disjoint convex compact sets in $\mathbb{C}^n$, then $X_1\cup X_2$ is $\mathscr{A}(\mathbb{C}^n)$-convex.*

Proof. We can take a linear polynomial as f in the Separation Lemma. ∎

By virtue of the Separation Lemma, it is easy to apply in $\mathbb{C}^n$ the arguments used to prove the theorems on universal functions in $\mathbb{C}$.

2. UNIVERSAL FUNCTIONS ON $\mathbb{C}^N$

In this section we improve the results in the previous paper [1] by the first author. We denote the usual norm of $z=(z_1, ..., z_n)\in\mathbb{C}^n$ by $\|z\|:=(\sum_{i=1}^n |z_i|^2)^{1/2}$. For $a\in\mathbb{C}^n$ and $r>0$, we define the open ball $B(a;r):=\{z\in\mathbb{C}^n;\ \|z-a\|<r\}$ centered at a with radius r. The closed ball centered at a with radius r is defined by $\overline{B(a;r)}:=\{z\in\mathbb{C}^n;\ \|z-a\|\leqslant r\}$. We write $B(r)=B(0;r)$ and $\overline{B(r)}=B(0;r)$ for the sake of simplicity. For a compact subset K of $\mathbb{C}^n$, we denote by $\mathscr{A}(K)$ the set of all functions which are holomorphic in some neighbourhood of K. We take a sequence $\{P_1, P_2, ...\}$ of holomorphic functions on $\mathbb{C}^n$ which is dense in $\mathscr{A}(\mathbb{C}^n)$.

THEOREM 1. *Let $\{a^{(j)}\}$ be a sequence in $\mathbb{C}^n$ with $\|a^{(j)}\|\to\infty$ $(j\to\infty)$, and let $\{\varepsilon^{(k)}\}$ be a decreasing sequence of positive numbers with $\varepsilon^{(k)}\to 0$ $(k\to\infty)$. Then there exist $F\in\mathscr{A}(\mathbb{C}^n)$ and an increasing sequence $\{r_k\}$ with $r_k\to\infty$ $(k\to\infty)$ having the following property. For any k, there exists j_k such that*

$$|F(z+a^{(j_k)})-P_k(z)|<\varepsilon^{(k)} \qquad \text{on } \overline{B(r_k)}.$$

Proof. There exists a subsequence $\{a^{(j_k)}\}$ of $\{a^{(j)}\}$ as follows. If we set $r_0:=1$ and

$$r_k:=\tfrac{1}{2}(\|a^{(j_k)}\|-\|a^{(j_{k-1})}\|),$$

then $\{r_k\}$ is an increasing sequence of positive numbers and $r_k\to\infty$ $(k\to\infty)$. We put $A:=\sum_{k=1}^{\infty}(1/k^2)$.

We construct a sequence $\{Q_k\}_0^\infty$ of polynomials by induction. Let $Q_0 \equiv 0$. Assume that Q_{k-1} has already been determined. We define $L_{k-1} := \overline{B(\|a^{(j_{k-1})}\| + r_{k-1})}$ and $C_k := \overline{B(a^{(j_k)}; r_k)}$. By the choice of $\{r_k\}$ we have $L_{k-1} \cap C_k = \emptyset$. Applying the Corollary, we obtain that $L_{k-1} \cup C_k$ is $\mathscr{A}(\mathbb{C}^n)$-convex. Then there exists a polynomial Q_k such that

$$\max_{z \in L_{k-1}} |Q_k(z) - Q_{k-1}(z)| < \frac{\varepsilon^{(k)}}{2k^2 A},$$

$$\max_{z \in C_k} |Q_k(z) - P_k(z - a^{(j_k)})| < \frac{\varepsilon^{(k)}}{2}$$

(see Theorem 4.3.2 in Hörmander [13]).

Define

$$F(z) := \sum_{k=1}^{\infty} (Q_k(z) - Q_{k-1}(z)).$$

Then $F(z)$ is an entire function on $\mathbb{C}^n$. Since $C_k \subset L_k$, we obtain

$$\begin{aligned}
&\max_{z \in \overline{B(r_k)}} |F(z + a^{(j_k)}) - P_k(z)| \\
&\quad = \max_{z \in C_k} |F(z) - P_k(z - a^{(j_k)})| \\
&\quad \leqslant \max_{z \in L_k} |F(z) - Q_k(z)| + \max_{z \in C_k} |Q_k(z) - P_k(z - a^{(j_k)})| \\
&\quad \leqslant \sum_{\mu = k}^{\infty} \max_{z \in L_k} |Q_{\mu+1}(z) - Q_\mu(z)| + \frac{\varepsilon^{(k)}}{2} \\
&\quad < \varepsilon^{(k)}. \quad \blacksquare
\end{aligned}$$

Let $\mathscr{K}$ be the set of all compact sets K in $\mathbb{C}^n$ with $\hat{K} = K$. In the same way as in [1], we can prove the following theorems, which are extensions of Birkhoff's theorem and Luh's theorem. We give their proofs for the sake of completeness.

THEOREM 2. *Let $\{a^{(j)}\}$ be a sequence in $\mathbb{C}^n$ with $\|a^{(j)}\| \to \infty$ $(j \to \infty)$. Then there is an entire function $F(z)$ on $\mathbb{C}^n$ with the property that for every entire function $f(z)$ there exists a subsequence $\{a^{(j_k)}\}$ of $\{a^{(j)}\}$ such that $\{F(z + a^{(j_k)})\}$ converges to $f(z)$ uniformly on compact sets in $\mathbb{C}^n$.*

Proof. Take a decreasing sequence $\{\varepsilon^{(k)}\}$ of positive numbers with $\varepsilon^{(k)} \to 0$ $(k \to \infty)$. For the given sequence $\{a^{(j)}\}$ and this sequence $\{\varepsilon^{(k)}\}$, there exist $F \in \mathscr{A}(\mathbb{C}^n)$ and an increasing sequence $\{r_k\}$ with $r_k \to \infty$ $(k \to \infty)$ in Theorem 1.

We show that this entire function F has the required property. Let f be an entire function on $\mathbb{C}^n$. Since $\{P_k\}$ is dense in $\mathscr{A}(\mathbb{C}^n)$, there exists a subsequence $\{P_{k_\ell}\}$ such that $f(z)=\lim_{\ell\to\infty} P_{k_\ell}(z)$. Then, for any $\varepsilon>0$ and any $r>0$, there exists a positive integer ℓ_0 such that

$$|f(z)-P_\ell(z)|<\frac{\varepsilon}{2} \quad \text{for} \quad z\in\overline{B(r)} \quad \text{and} \quad \ell\geqslant\ell_0.$$

We can take a positive integer ℓ_1 with $\ell_1\geqslant\ell_0$ such that $\varepsilon^{(k_{\ell_1})}<\varepsilon/2$ and $r_{k_{\ell_1}}>r$. For $z\in\overline{B(r)}$ and $\ell\geqslant\ell_1$, we have

$$\begin{aligned}|F(z+a^{(j_{k_\ell})})-f(z)| &\leqslant |F(z+a^{(j_{k_\ell})})-P_{k_\ell}(z)|+|P_{k_\ell}(z)-f(z)|\\ &<\varepsilon^{(k_\ell)}+\frac{\varepsilon}{2}<\varepsilon.\end{aligned}$$

This completes the proof. ∎

THEOREM 3. *Let $\{a^{(j)}\}$ be a sequence in $\mathbb{C}^n$ with $\|a^{(j)}\|\to\infty$ $(j\to\infty)$. Then there is an entire function $F(z)$ on $\mathbb{C}^n$ with the property that for every $K\in\mathscr{K}$ and every $f\in\mathscr{A}(K)$ there exists a subsequence $\{a^{(j_k)}\}$ of $\{a^{(j)}\}$ such that $\{F(z+a^{(j_k)})\}$ converges to $f(z)$ uniformly on K.*

Proof. Let $\{\varepsilon^{(k)}\}$ be a decreasing sequence of positive numbers with $\varepsilon^{(k)}\to 0$. We can take $F\in\mathscr{A}(\mathbb{C}^n)$ and an increasing sequence $\{r_k\}$ as in the proof of Theorem 2. For $K\in\mathscr{K}$ and $f\in\mathscr{A}(K)$, there exists a subsequence $\{P_{k_\ell}\}$ of $\{P_k\}$ such that

$$\max_{z\in K}|f(z)-P_{k_\ell}(z)|<\frac{1}{\ell}$$

for any positive integer ℓ by an approximation theorem in several variables (Corollary 5.2.9 in Hörmander [13]). If ℓ is sufficiently large, then $K\subset\overline{B(r_{k_\ell})}$. Therefore we have

$$\begin{aligned}&\max_{z\in K}|F(z+a^{(j_{k_\ell})})-f(z)|\\ &\qquad\leqslant \max_{z\in\overline{B(r_{k_\ell})}}|F(z+a^{(j_{k_\ell})})-P_{k_\ell}(z)|+\max_{z\in K}|f(z)-P_{k_\ell}(z)|\\ &\qquad<\varepsilon^{(k_\ell)}+\frac{1}{\ell}.\end{aligned}$$

Hence we obtain the conclusion. ∎

3. UNIVERSAL FUNCTIONS ON $GL(N, \mathbb{C})$

We denote by $M(n, \mathbb{C})$ the set of all square matrices of degree n with complex coefficients. The group $GL(n, \mathbb{C}) = \{Z \in M(n, \mathbb{C}); \det Z \neq 0\}$ is called the general linear group of degree n over $\mathbb{C}$. For the sake of simplicity, we write $G = GL(n, \mathbb{C})$. We define for $r > 0$,

$$C(r) := \left\{ Z \in G; \frac{1}{r} \leqslant |\det Z| \leqslant r \right\},$$

$$D(r) := \{ Z = (z_{ij}) \in M(n, \mathbb{C}); |z_{ij}| \leqslant r \ (i, j = 1, ..., n) \},$$

$$K(r) := C(r) \cap D(r).$$

Then $K(r)$ is a compact set in G with $K(r)\hat{} = K(r)$, where $K(r)\hat{}$ is the $\mathscr{A}(G)$-hull of $K(r)$.

According to [26], we define two families $B(\mathbb{C})$ and $B(\mathbb{C}^*)$ of compact sets in $\mathbb{C}$ and $\mathbb{C}^*$, respectively. The set $B(\mathbb{C})$ consists of compact sets in $\mathbb{C}$ whose complement are connected. Similarly, $B(\mathbb{C}^*)$ is the set of all compact sets K in $\mathbb{C}^*$ such that $\mathbb{C}^* \backslash K$ is connected. It is obvious that $B(\mathbb{C}^*) \subset B(\mathbb{C})$.

LEMMA 1. *There exists a sequence $\mathscr{F} = \{L_\ell\}$ in $B(\mathbb{C})$ such that for any $L \in B(\mathbb{C})$ and any neighbourhood U of L there exists L_ℓ with $L \subset L_\ell \subset U$.*

Proof. Let $\mathscr{B} = \{B_i\}_{i \in I}$ be a countable open base of $\mathbb{C}$ which consists of simply connected relatively compact open sets. Take a sequence $\{K_j\}$ in $B(\mathbb{C})$ such that $K_j \subset K_{j+1}$ and $\bigcup_{j=1}^{\infty} K_j = \mathbb{C}$.

Consider $L \in B(\mathbb{C})$ and a neighbourhood W of L. Since every compact set in $B(\mathbb{C})$ has a fundamental neighbourhood system consisting of simply connected neighbourhoods (Lemma 1 in [26]), there exist simply connected neighbourhoods U, V of L and $j_0 \in \mathbb{N}$ such that

$$L \subset \bar{V} \subset U \subset W, \qquad U \subset K_{j_0}.$$

The compact set $K_{j_0} \backslash U$ can be covered by a finite number of open sets $B_{i_1}, ..., B_{i_s} \in \mathscr{B}$ such that $B_{i_k} \subset \mathbb{C} \backslash \bar{V}$ and $B_{i_k} \cap (K_{j_0} \backslash U) \neq \varnothing$ for $k = 1, ..., s$.

We set $M := K_{j_0} \backslash (\bigcup_{k=1}^{s} B_{i_k})$. Then we have $L \subset M \subset U$. Since $\mathbb{C} \backslash M$ is connected, $M \in B(\mathbb{C})$.

Let $\mathscr{F}$ be the set of all compact sets in $B(\mathbb{C})$ which are of the form $K_j \backslash (\bigcup_{i' \in I'} B_{i'})$, where I' is a finite set. Then $\mathscr{F}$ is denumerable and has the required properties. ∎

DEFINITION. We denote by $B(G)$ the set of all compact subsets K of G with $\hat{K}=K$ such that there exists $f\in\mathscr{A}(M(n,\mathbb{C}))$ with $f(0)\notin(f(K))^\wedge$, where $(f(K))^\wedge$ is the $\mathscr{A}(\mathbb{C})$-hull of $f(K)$.

PROPOSITION 1. *There exists a sequence $\mathscr{F}=\{K_i\}$ in $B(G)$ such that for any $K\in B(G)$ there exists K_i with $K\subset K_i$.*

Proof. Let $\varphi(Z):=\mathrm{Trace}({}^t\bar{Z}Z)+1/|\det Z|^2$. Then φ is a C^∞ strictly plurisubharmonic function on G and satisfies that $\{Z\in G;\ \varphi(Z)<c\}$ is a relatively compact subset of G for any $c>0$. Let $\{c_\alpha\}$ be a strictly increasing sequence of positive numbers such that $c_\alpha\to\infty$ $(\alpha\to\infty)$. Take a dense countable subset $\{f_\lambda\}_1^\infty\subset\mathscr{A}(M(n,\mathbb{C}))$. Let $\{L_\ell\}$ be a sequence in $B(\mathbb{C})$ in Lemma 1. For any $f\in\mathscr{A}(M(n,\mathbb{C}))$, we denote by $\{L_{\ell_j}(f)\}$ the subsequence of $\{L_\ell\}$ consisting of L_ℓ with $f(0)\notin L_\ell$.

We define

$$K^\alpha_{\ell_j}(f):=f^{-1}(L_{\ell_j}(f))\cap\{Z\in G;\ \varphi(Z)\leqslant c_\alpha\}.$$

Then $K^\alpha_{\ell_j}(f)$ is a compact subset and satisfies $(K^\alpha_{\ell_j}(f))^\wedge=K^\alpha_{\ell_j}(f)$. We set

$$\mathscr{F}:=\{K^\alpha_{\ell_j}(f_\lambda);\ \alpha,\lambda\in\mathbb{N}\}.$$

Then $\mathscr{F}$ is a countable set.

We show that the family $\mathscr{F}$ has the desired property. For any $K\in B(G)$, there exists $f\in\mathscr{A}(M(n,\mathbb{C}))$ such that $f(0)\notin(f(K))^\wedge$. Since $(f(K))^\wedge\in B(\mathbb{C})$, $(f(K))^\wedge$ has two relatively compact simply connected neighbourhoods U_1 and U_2 such that $\overline{U_1}\subset U_2$, $f(0)\notin\overline{U_2}$. By the density of $\{f_\lambda\}$ in $\mathscr{A}(M(n,\mathbb{C}))$, we can take f_λ such that $f_\lambda(K)\subset U_1$, $f_\lambda(0)\notin\overline{U_2}$. Since $(\widehat{\overline{U_1}})\subset U_2$, there exists L_ℓ such that $(\widehat{\overline{U_1}})\subset L_\ell\subset U_2$ by Lemma 1. It is obvious that $(f_\lambda(K))^\wedge\subset(\widehat{\overline{U_1}})$. Then we have $K^\alpha_\ell(f_\lambda)\supset K$ for a sufficiently large α. ∎

LEMMA 2. *For any finite number of compact sets $K_1, K_2, ..., K_s\in B(G)$ and any $r>0$, there exist $C_1, C_2, ..., C_s\in G$ such that $(K(r)\cup(\bigcup_{i=1}^s C_iK_i))^\wedge=K(r)\cup(\bigcup_{i=1}^s C_iK_i)$, $K(r)\cap C_iK_i=\varnothing$ for $i=1, ..., s$ and $C_iK_i\cap C_jK_j=\varnothing$ for $i\neq j$.*

Proof. We prove the lemma by induction.

Let $s=1$. For any $K\in B(G)$ there exists $f\in\mathscr{A}(M(n,\mathbb{C}))$ such that $f(0)\notin(f(K))^\wedge$. Take a closed ball $\bar{B}$ with center at $f(0)$ such that $\bar{B}\cap(f(K))^\wedge=\varnothing$. Let $r>0$. Then there exists $c\in\mathbb{C}^*$ such that if we set $C^{-1}=c^{-1}I$, then $f(C^{-1}K(r))\subset\bar{B}$ and $K(r)\cap CK=\varnothing$. Here I is the unit matrix of degree n. We define

$$f_C(Z):=f(C^{-1}Z),\qquad Z\in M(n,\mathbb{C}).$$

Then f_C is a holomorphic function on $M(n, \mathbb{C})$, and has the properties that $f_C(CK) = f(K)$, $f_C(K(r)) \subset \bar{B}$. Hence we have

$$(f_C(K(r)))\hat{} \cap (f_C(CK))\hat{} \subset \bar{B} \cap (f(K))\hat{} = \varnothing.$$

Therefore, we obtain by the Separation Lemma

$$(K(r) \cup CK)\hat{} = K(r)\hat{} \cup (CK)\hat{} = K(r) \cup CK.$$

Let $s \geqslant 2$. Suppose that the lemma is proved for $s-1$ compact sets in $B(G)$, and $C_1, C_2, ..., C_{s-1} \in G$ with the required properties have been obtained. Take $t > 0$ such that

$$K(t) \supset \left(K(r) \cup \left(\bigcup_{i=1}^{s-1} C_i K_i \right) \right).$$

Applying the argument used in the case $s = 1$ to $K(t)$ and K_s, we get C_s with the required properties. ∎

THEOREM 4. *There exists a holomorphic function F on G such that for any $K \in B(G)$, for any $f \in \mathscr{A}(K)$ and for any $\varepsilon > 0$, there exists $C \in G$ with*

$$\max_{Z \in K} |F(CZ) - f(Z)| < \varepsilon.$$

Proof. Let $\mathscr{F} = \{K_i\}_1^\infty$ be a sequence in $B(G)$ as in Proposition 1, and let $\{f_\lambda\}_1^\infty$ be a countable dense subset of $\mathscr{A}(M(n, \mathbb{C}))$ as in the proof of Proposition 1.

Take r_0 with $K_1 \subset K(r_0)$. By Lemma 2 there exists $C_{1,1} \in G$ such that $(K(r_0) \cup C_{1,1} K_1)\hat{} = K(r_0) \cup C_{1,1} K_1$ and $K(r_0) \cap C_{1,1} K_1 = \varnothing$. We can choose $r_1 > r_0$ such that $C_{1,1} K_1 \subset K(r_1)$ and $K_2 \subset K(r_1)$. Inductively, we can take sequences $\{r_i\}$ with $0 < r_0 < r_1 < \cdots < r_i < \cdots$ and $\{C_{s,i}\}_{1 \leqslant s \leqslant i, 1 \leqslant i \leqslant \infty} \subset G$ such that

(1) $K_i \subset K(r_{i-1})$,

(2) $$\left(K(r_{i-1}) \cup \left(\bigcup_{s=1}^{i} C_{s,i} K_s \right) \right)^{\wedge} = K(r_{i-1}) \cup \left(\bigcup_{s=1}^{i} C_{s,i} K_s \right)$$

(3) $K(r_{i-1}) \cap C_{s,i} K_s = \varnothing, \qquad 1 \leqslant s \leqslant i,$

(4) $C_{s,i} K_s \cap C_{t,i} K_t = \varnothing, \qquad s \neq t$

(5) $C_{s,i} K_s \subset K(r_i), \qquad 1 \leqslant s \leqslant i.$

By the first condition we have $\bigcup_{i=0}^{\infty} K(r_i) = \bigcup_{i=1}^{\infty} K_i = G$. Therefore, we can define by induction a sequence $\{g_i\}_0^{\infty}$ of holomorphic functions on G such that

$$(1) \quad g_0(Z) \equiv 0,$$

$$(2) \quad \max_{Z \in K(r_{i-1})} |g_i(Z) - g_{i-1}(Z)| < \frac{1}{i^2},$$

$$(3) \quad \max_{Z \in C_{s,i} K_s} |g_i(Z) - f_i(C_{s,i}^{-1} Z)| < \frac{1}{i^2}, \qquad 1 \leqslant s \leqslant i.$$

We set

$$F := \sum_{i=1}^{\infty} (g_i - g_{i-1}).$$

Then the series in the right-hand side converges uniformly on compact sets in G, hence $F \in \mathscr{A}(G)$. We have for $1 \leqslant s \leqslant i$,

$$\begin{aligned}
&\max_{Z \in K_s} |F(C_{s,i} Z) - f_i(Z)| \\
&\quad = \max_{W \in C_{s,i} K_s} |F(W) - f_i(C_{s,i}^{-1} W)| \\
&\quad \leqslant \max_{W \in C_{s,i} K_s} |F(W) - g_i(W)| + \max_{W \in C_{s,i} K_s} |g_i(W) - f_i(C_{s,i}^{-1} W)| \\
&\quad < \sum_{v=i}^{\infty} \max_{W \in K(r_i)} |g_{v+1}(W) - g_v(W)| + \frac{1}{i^2} \\
&\quad < \sum_{v=i}^{\infty} \frac{1}{v^2} + \frac{1}{i^2} = \eta(i) \qquad \text{with} \quad \lim_{i \to \infty} \eta(i) = 0.
\end{aligned}$$

We show that F has the required property. Let $K \in B(G)$, $f \in \mathscr{A}(K)$, and $\varepsilon > 0$. There exists s such that $K \subset K_s$. Choose $i \geqslant s$ such that $\eta(i) < \varepsilon/2$ and

$$\max_{Z \in K} |f(Z) - f_i(Z)| < \frac{\varepsilon}{2}.$$

Then we obtain

$$\begin{aligned}
\max_{Z \in K} |F(C_{s,i} Z) - f(Z)| &\leqslant \max_{Z \in K_s} |F(C_{s,i} Z) - f_i(Z)| + \max_{Z \in K} |f(Z) - f_i(Z)| \\
&< \eta(i) + \frac{\varepsilon}{2} < \varepsilon. \quad \blacksquare
\end{aligned}$$

4. REMARKS

It is difficult in general to determine which compact set belongs to $B(G)$. However, we know that $B(G)$ contains the following type of compact sets.

Suppose that the matrix 0 does not belong to the $\mathscr{A}(M(n,\mathbb{C}))$-hull of a compact set $K=\hat{K}\subset G$. Then there exists $f\in\mathscr{A}(M(n,\mathbb{C}))$ such that $|f(0)|>\max_{Z\in K}|f(Z)|$. Therefore $f(0)\notin(f(K))^\wedge$. Hence we have $K\in B(G)$. In particular, if $n=1$, we have $B(G)=B(\mathbb{C}^*)$. Then Theorem 4 contains the result in [26] due to the second author.

We also give a counterexample. Consider the mapping $i\colon \mathbb{C}\hookrightarrow M(n,\mathbb{C})$ defined by $i(c)=cI$. We set $N:=i(\mathbb{C}^*)$ and $K:=i(S^1)$, where S^1 is the unit circle. Then K is $\mathscr{A}(N)$-convex. Since N is a closed submanifold of a Stein manifold G, the restriction mapping from $\mathscr{A}(G)$ to $\mathscr{A}(N)$ is surjective. So we see that K is $\mathscr{A}(G)$-convex.

Suppose that $K\in B(G)$. Then there exists $f\in\mathscr{A}(M(n,\mathbb{C}))$ such that $f(0)\notin(f(K))^\wedge$. Therefore we can take a holomorphic function g on $\mathbb{C}$ such that $|g(f(0))|>\max_{\zeta\in(f(K))^\wedge}|g(\zeta)|$. However, we have

$$|(g\circ f)(0)|=|(g\circ f\circ i)(0)|\leqslant\max_{c\in S^1}|(g\circ f\circ i)(c)|=\max_{Z\in K}|(g\circ f)(Z)|.$$

This is a contradiction. Then $K\notin B(G)$.

Finally we remark that Birkhoff's theorem and Luh's theorem can be proved as a corollary of each other by Mergelyan's theorem (of course Birkhoff's theorem is almost 50 years older). But a theorem analogous to Birkhoff's for the multiplicative group $\mathbb{C}^*$ does not hold. We can see this fact by the argument in Remark 2 in [26]. While for $n=1$ it is not possible to find a universal function for all $\mathscr{A}(\mathbb{C}^*)$-convex compact sets K, we do not know whether such a function exists for G in the case $n>1$. If this question has a positive answer, then a theorem similar to Birkhoff's one will be true in G for $n>1$.

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